9.1 Geometric Sequences

Objectives: 1. Recognize and extend geometric sequences.
2. Find the nth term of a geometric sequence.

The table shows the heights of a bungee jumper's bounces.

Bounce	1	2	3
Height (ft)	200	80	32

The height of the bounces shown in the table above form a *geometric sequence.* In a geometric sequence, the ratio of successive terms is the same number r, called the common ratio.

Geometric sequences can be thought of as functions. The term number, or position in the sequence, is the input, and the term itself is the output.

1	2	3	4	← Position
↓	↓	↓	↓	
3	6	12	24	← Term
a_1	a_2	a_3	a_4	

Finding a Term of a Geometric Sequence

The nth term of a geometric sequence with **common ratio** r is

$$a_n = a_{n-1}r$$

Find the next three terms in the geometric sequence.

1, 4, 16, 64,... $\times 4$, $\times 4$, $\times 4$ 256, 1024, 4096

$\frac{64}{16} = 4$ $r = 4$

$\frac{16}{4} = 4$

$\frac{4}{1} = 4$

$$-9, 3, -1, \frac{1}{3}, -\frac{1}{9}$$

$\frac{3}{9} = \frac{1}{3}$

$r = -\frac{1}{3}$

$\frac{1}{27}$, $-\frac{1}{81}$, $\frac{1}{243}$

5, −10, 20, −40,...

$r = -2$

80, −160, 320

512, 384, 288,...

$r = \frac{384}{512}$

$r = .75$

216, 162, 121.5

If the first term of a geometric sequence is a_1, the nth term is a_n , and the common ratio is r, then

$$a_n = a_1 r^{n-1}$$

terms asking for

*n*th term 1st term Common ratio

The first term of a geometric sequence is 500, and the common ratio is 0.2. What is the 7th term of the sequence?

$a_n = a_1 r^{n-1}$ $n = 7$

$a_7 = 500(.2)^{7-1}$ $a_1 = 500$

$a_7 = 500(.2)^6$ $r = .2$

$\boxed{a_7 = \frac{4}{125} \quad a_7 = .032}$

For a geometric sequence, a_1 = 5, and r = 2. Find the 6th term of the sequence.

$a_6 = 5(2)^{6-1}$

$5(2)^5$

$5(32)$

160

What is the 9th term of the geometric sequence 2, −6, 18, −54, ...?

$a_n = a_1 r^{n-1}$

$n = 9$

$a_1 = 2$

$r = -3$

$a_9 = 2(-3)^{9-1}$

$a_9 = 2(-3)^8$

$a_9 = 2(6561)$

$\boxed{a_9 = 13{,}122}$

What is the 8th term of the sequence 1000, 500, 250, 125, ...?

$a_8 = 1000(.5)^{8-1}$ $\frac{500}{1000} = \frac{1}{2} = r$

$a_8 = 1000(.5)^7$

$a_8 = 7.8125$

A ball is dropped from a tower. The table shows the heights of the balls bounces, which form a geometric sequence. What is the height of the 6th bounce?

$a_n = a_1 r^{n-1}$ $n = 6$

$r = \frac{150}{300} = \frac{1}{2}$

$a_6 = 300(\frac{1}{2})^{6-1}$

$a_1 = 300$

$300(\frac{1}{2})^5$

$\boxed{a_6 = 9.375 \text{ cm}}$

Bounce	Height (cm)
1	300
2	150
3	75

The table shows a car's value for 3 years after it is purchased. The values form a geometric sequence. How much will the car be worth in the 10th year?

$a_n = a_1 r^{n-1}$ $n = 10$

$a_{10} = 10000(.8)^{10-1}$ $r = .8$

$10000(.8)^9$ $a_1 = 10{,}000$

1342.17728

$\boxed{a_{10} = \$1{,}342.18}$

Year	Value ($)
1	10,000
2	8,000
3	6,400

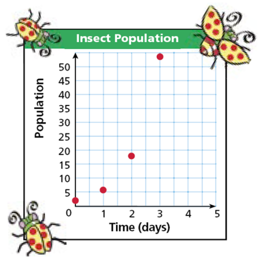
9.2 Exponential Functions

Objectives: 1. Evaluate exponential functions.
2. Identify and graph exponential functions.

The table and the graph show an insect population that increases over time.

Time (days)	Population
0	2
1	6
2	18
3	54

$\times 3$
$\times 3$
$\times 3$



A function rule that describes the pattern above is $f(x)=2(3)^x$. This type of function, in which the independent variable appears in an exponent, is an **exponential function**. Notice that 2 is the starting population and 3 is the amount by which the population is multiplied each day.

Exponential Functions

An exponential function has the form $f(x) = ab^x$, where $a \neq 0$, $b \neq 1$, and $b > 0$.

The function $f(x) = 500(1.035)^x$ models the amount of money in a certificate of deposit after x years. How much money will there be in 6 years?

$f(6) = 500(1.035)^6$

$f(6) = 614.6276632$

$f(6) = \$614.63$

The function $f(x) = 200{,}000(0.98)^x$, where x is the time in years, models the population of a city. What will the population be in 7 years?

$f(7) = 200{,}000(.98)^7$

$f(7) = 173{,}625.1066$

$\approx 173{,}625$

The function $f(x) = 8(0.75)^x$ models the width of a photograph in inches after it has been reduced by 25% x times. What is the width of the photograph after it has been reduced 3 times?

$100 - 25 = .75$

$f(3) = 8(.75)^3$

$f(3) = 3.375$ in.

Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

{(0, 4), (1, 12), (2, 36), (3, 108)}

Yes, the y is constant rate of multiplying by 3.

$f(x)=4(3)^x$

X	Y
0	4
1	12
2	36
3	108

+1 ×3 ×3 ×3

{(−1, −64), (0, 0), (1, 64), (2, 128)}

No, because you are adding 64 not multiplying

X	Y
-1	-64
0	0
1	64
2	128

+1 +64 +64 +64

{(−1, 1), (0, 0), (1, 1), (2, 4)}

No, because the y-values aren't multiplied.

X	Y
-1	1
0	0
1	1
2	4

+1 +1 +1 +3

{(−2, 4), (−1 , 2), (0, 1), (1, 0.5)}

Yes because you multiply by ½ each time.

$f(x)=1(\frac{1}{2})^x$

X	Y
-2	4
-1	2
0	1
1	.5

+1 $\times\frac{1}{2}$ $\times\frac{1}{2}$ $\times\frac{1}{2}$

To graph an exponential function, choose several values of *x* (positive, negative, and 0) and generate ordered pairs. Plot the points and connect them with a smooth curve.

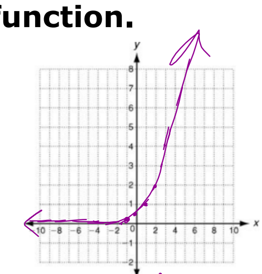
Graph each function.

$y = 0.5(2)^x$

X	Y
-1	.25
0	.5
1	1
2	2
3	4

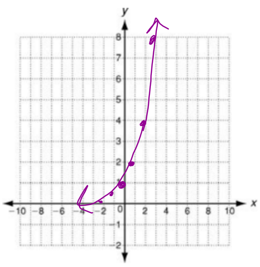
$\frac{1}{2}(2)^{-1}$

$\frac{1}{2}(\frac{1}{2})$



$y = 2^x$

X	Y
-2	1/4
-1	1/2
0	1
1	2
2	4

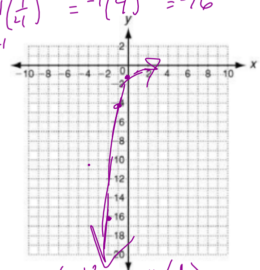


$y=-1(\frac{1}{4})^{-2} = -1(4)^2 = -16$

$y = -1\left(\frac{1}{4}\right)^x$

$y=-1(\frac{1}{4})^{-1}$

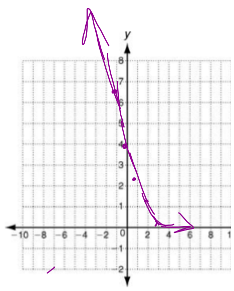
X	Y
-2	-16
-1	-4
0	-1
1	-1/4
2	-1/16



$y=-1(\frac{1}{4})^2 = -1(\frac{1}{16})$

$y = 4(0.6)^x$

X	Y
-2	11.1
-1	6.67
0	4
1	2.4
2	1.44
3	.8



9.3 Exponential Growth and Decay

Objective: 1. Solve problems involving exponential growth and decay.

exponential Growth occurs when an quantity increases by the same rate *r* in each period *t*. When this happens, the value of the quantity at any given time can be calculated as a function of the rate and the original amount.

$y = a(1 + r)^t$ where $a > 0$

y = final amount
a = original amount
r = rate of growth (decimal)
t = time

% → decimal you divide by 100

The original value of a painting is $9,000 and the value increases by 7% each year. Write an exponential growth function to model this situation. Then find the painting's value in 15 years.

$y = a(1+r)^t$

$y = 9000(1+.07)^{15}$

$y = 9000(1.07)^{15}$

$y = 24831.28387$

$y = \$24831.28$

A sculpture is increasing in value at a rate of 8% per year, and its value in 2000 was $1200. Write an exponential growth function to model this situation. Then find the sculpture's value in 2006.

$y = a(1+r)^t$

$y = 1200(1+.08)^6$

$y = 1200(1.08)^6$

$y = \$1904.25$

A common application of exponential growth is *compound interest.* Recall that simple interest is earned or paid only on the principal. Compound Interest is interest earned or paid on *both* the principal and previously earned interest.

$A = P(1 + \frac{r}{n})^{nt}$

A = balance after t years
P = Principal or original amount
r = interest rate (decimal)
n = # of times compounded/year
t = time

monthly — 12
quarterly — 4
(yearly) anually — 1
Semi-anually — 2

Write a compound interest function to model each situation. Then find the balance after the given number of years.

$1200 invested at a rate of 2% compounded quarterly; 3 years

4

$A = P(1+\frac{r}{n})^{nt}$ $A = 1200\left(1+\frac{.02}{4}\right)^{4(3)} = 1200(1+.005)^{12} = 1200(1.005)^{12} = \$1,274.01$

$15,000 invested at a rate of 4.8% compounded monthly; 2 years.

$A = 15000\left(1+\frac{.048}{12}\right)^{12(2)}$

$15000(1+.004)^{24}$

$1500(1.004)^{24}$

$A = \$16,508.22$

Exponential Decay occurs when a quantity decreases by the same rate *r* in each time period *t*. Just like exponential growth, the value of the quantity at any given time can be calculated by using the rate and the original amount.

$y = a(1 - r)^t$ where $a > 0$

- y = final amount
- a = original amount
- r = rate of decay (decimal)
- t = time

The population of a town is decreasing at a rate of 3% per year. In 2000 there were 1700 people. Write an exponential decay function to model this situation. Then find the population in 2012.

$y = a(1 - r)^t$

$1700(1 - .03)^{12}$

$1700(.97)^{12}$

$y = 1179.53$

≈ 1180 people

The fish population in a local stream is decreasing at a rate of 5% per year. The original population was 48,000. Write an exponential decay function to model this situation. Then find the population after 7 years.

$y = a(1 - r)^t$

$y = 48000(1 - .05)^7$

$y = 48000(.95)^7$

$y = 33520.19$

$\boxed{\approx 33{,}520 \text{ fish}}$

A common application of exponential decay is *half-life.* The half-life of a substance is the time it takes for one-half of the substance to decay into another substance.

$A = P(0.5)^t$

- A = final amount
- P = original amount
- t = # of half-lives in a given time period

Astatine-218 has a half-life of 2 seconds.

Find the amount left from a 500 gram sample of astatine-218 after 10 seconds.

$t = \frac{10}{2} = 5$

$A = P(.5)^t$

$= 500(.5)^5$

$A = 15.625$ g

Find the amount left from a 500-gram sample of astatine-218 after 1 minute.

$t = \frac{60 \text{ sec}}{2 \text{ sec}} = 30$

$A = 500(.5)^{30}$

$\boxed{A = .00000047 \text{ g}}$

$\boxed{4.7 \times 10^{-7}}$ g

9.4 Linear, Quadratic, and Exponential Models

Objectives: 1. Compare linear, quadratic, and exponential models.
2. Given a set of data, decide which type of function models the data and write an equation to describe the data.

Look at the tables and graphs below. The data show three ways you have learned that variable quantities can be related.

The relationship shown is linear.

Both x and y-values have a constant rate added or subtracted.

Training Heart Rate

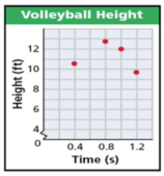
Age (yr)	Beats/min
20	170
30	161.5
40	153
50	144.5

The relationship shown is Quadratic.

The second difference y-value is added or subtracted at a constant rate.

Volleyball Height

Time (s)	Height (ft)
0.4	10.44
0.8	12.76
1	12
1.2	9.96



The relationship shown is Exponential.

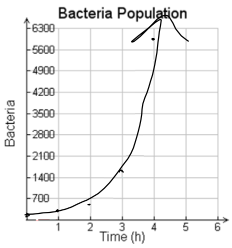
The y-value is multiplied at a constant rate.

Volleyball Tournament

Round	Teams Left
1	16
2	8
3	4
4	2

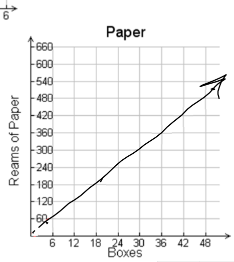
Graph each data set. Which kind of model best describes the data?

Time(h)	Bacteria
0	24
1	96
2	384
3	1536
4	6144



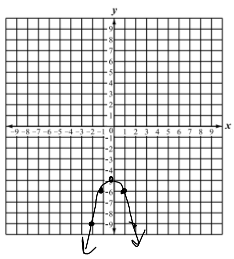
Exponential

Boxes	Reams of paper
1	10
5	50
20	200
50	500



Linear

x	y
−3	−14
−2	−9
−1	−6
0	−5
1	−6
2	−9
3	−14



Quadratic

Look for a pattern in each data set to determine which kind of model best describes the data.

Height of golf ball	
Time (s)	**Height (ft)**
0	4
1	68
2	100
3	100
4	68

+64, +32, +0, −32

−32, −32, −32

Constant

Quadratic

Money in CD	
Time (yr)	Amount ($)
0	1000.00
1	1169.86
2	1368.67
3	1601.04

×1.16986

+169.86, +198.81, +232.37

+28.95, +33.56

Exponential

Data (1)	Data (2)
−2	10
−1	1
0	−2
1	1
2	10

−9, −3, +3, +9

+6, +6, +6

Quadratic

After deciding which model best fits the data, you can write a function. Recall the general forms of linear, quadratic, and exponential functions.

General Forms of Functions

LINEAR	QUADRATIC	EXPONENTIAL
$y = mx + b$	$y = ax^2 + bx + c$	$y = ab^x$

Use the data in the table to describe how the number of people changes. Then write a function that models the data. Use your function to predict the number of people who received the e-mail after one week.

E-mail forwarding	
Time (Days)	**Number of People Who Received the E-mail**
0	8
1	56
2	392
3	2744

×7, ×7, ×7

Exponential

$y = ab^x$

$y = 8(7)^x$

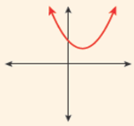
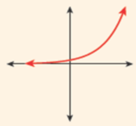
$y = 8(7)^7$

$y = 6{,}588{,}344$ people

9.5 Comparing Functions

Objective: 1. Compare functions in different representations and estimate rate of change.

You have studied different types of functions and how they can be represented as equations, graphs, and tables. Below is a review of three types of functions and some of their key properties.

	Linear	Quadratic	Exponential
Equation	$y = mx + b$ Example: $y = 2x + 1$	$y = ax^2 + bx + c$, $a \neq 0$ Example: $y = x^2 - 2x + 3$	$y = ab^x$, $a \neq 0$, $b \neq 1$, $b > 0$ Example: $y = 0.5(2)^x$
Graph			
Table	x: 0, 1, 2, 3, 4 y: 1, 3, 5, 7, 9 +2, +2, +2, +2 Constant first differences	x: 0, 1, 2, 3, 4 y: 3, 2, 3, 6, 11 −1, +1, +3, +5 +2, +2, +2 Constant second differences	x: 0, 1, 2, 3, 4 y: 0.5, 1, 2, 4, 8 ×2, ×2, ×2, ×2 Constant ratios

Sonia and Jackie each bake and sell cookies after school, and they each charge a delivery fee. The revenue for the sales of various numbers of cookies is shown. Compare the girls' prices by finding and interpreting the slopes and *y-intercepts*.

Sonia

$\frac{17-11}{48-24} = \frac{6}{24} = \$.25$/per cookie

Jackie's

$\frac{8-5}{20-10} = \frac{3}{10} = \$.30$/per cookie

Jackie has a higher cost per cookie but a cheaper delivery fee than Sonia.

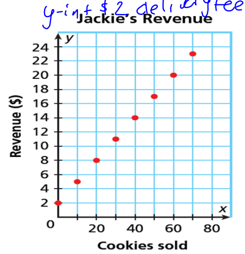
$5 delivery fee

Sonia's Revenue (y-int: 0)

Cookies Sold	Revenue ($)
24	11.00
48	17.00
72	23.00
96	29.00
120	35.00
144	41.00
168	47.00

24 $6

y-int $2 delivery fee



Dave and Arturo each deposit money into their checking accounts weekly. Their account information for the past several weeks is shown. Compare the accounts by finding and interpreting slopes and y-intercepts.

Dave Slope:

$12

$\frac{42-30}{1-0} = \frac{12}{1}$ deposits

Arturo slope

$\frac{32-24}{1-0} = \frac{8}{1}$

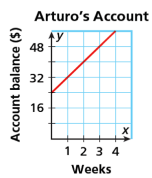
$8

y-int $30 starting

y-int $24

Dave's Account

Weeks	0	1	2	3
Account Balance ($)	30	42	54	66



They won't equal because Dave started with more and deposits more each week.

An investment analyst offers two different investment options for her customers. Compare the investments by finding and interpreting the average rates of change from year 0 to year 10.

$\frac{65-9}{10-0} = \frac{56}{10} = \$5.60/\text{year}$ A

$\frac{66.50-9}{10-0} = \frac{57.50}{10} = \$5.75$ B

Investment B offers more money per year.

x 2

[0, 10]

Investment B

Years	Value ($)
0	9.00
2	13.42
4	20.02
6	29.88
8	44.57
10	66.50
12	99.20

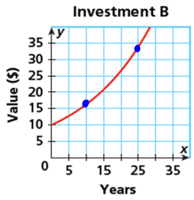
Compare the same investments' average rates of change from year 10 to year 25.

A) $\frac{42.92-17.91}{25-10} = \frac{25.01}{15} = \$1.67$

B) $\frac{34-16}{25-10} = \frac{18}{15} = \$1.20$

Investment A

Years	Value ($)
0	10.00
5	13.38
10	17.91
15	23.97
20	32.07
25	42.92



Investment A offers more money per year.